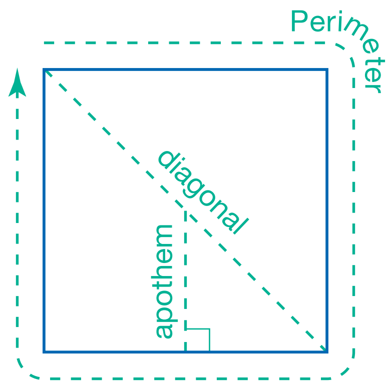
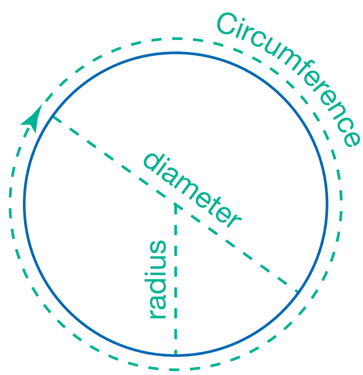
LESSON 16 Circumference and Area of Circles

UNDERSTAND Compare a **circle** to a square or other regular polygon. The shapes have corresponding characteristics and parts. For example, a diagonal of a square extends across the figure and through its center, much like a **diameter** of a circle. An **apothem** connects the center of a regular polygon to its side, and a **radius** connects the center of a circle to a point on the circle. Perimeter is the distance around a polygon, and **circumference** is the distance around a circle.



UNDERSTAND Line segments and perimeters are measured in linear units, such as meters (m) or feet (ft). So, when a circle is dilated by a factor of 2, its diameter, its radius, and its circumference all double, or increase by a factor of 2. Because these measurements change by the same factor, they are proportional to one another. For example, the circumference is directly proportional to the diameter, and the constant of proportionality is π.

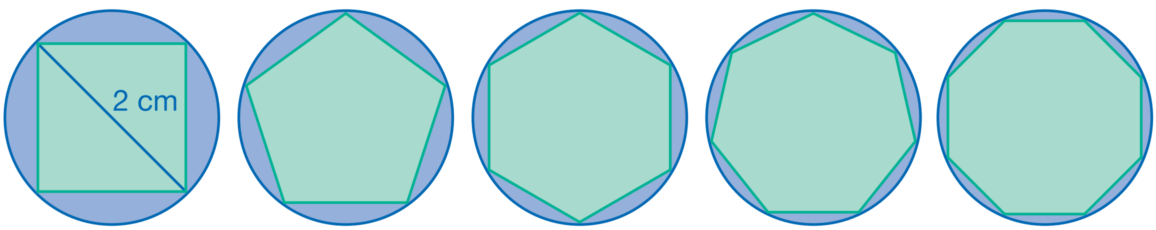
$$\frac{C_1}{d_1} = \frac{C_2}{d_2} \qquad \frac{C}{d} = \pi$$



This relationships leads to the formula for circumference, $C = \pi d$.

UNDERSTAND Look at the figures below. Notice how the sides of the polygons become closer to the curve of the circle as the number of sides increases. The perimeters of the polygons become better and better approximations of the circumference of the circle.

$C \approx 6.28$ cm $\quad A \approx 3.14$ cm^2



$P \approx 5.66$ cm	$P \approx 5.88$ cm	$P = 6$ cm	$P \approx 6.07$ cm	$P \approx 6.12$ cm
$A = 2$ cm^2	$A \approx 2.38$ cm^2	$A \approx 2.60$ cm^2	$A \approx 2.74$ cm^2	$A \approx 2.83$ cm^2

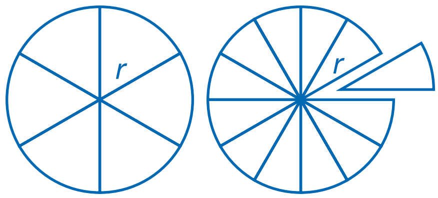
Notice also how, as the number of sides increases, the polygons cover more and more of the circle's area. The areas of the polygons become better and better approximations of the area of the circle.

Connect

Derive the formula for the area of a circle by dissecting a circle and rearranging its slices.

1 Dissect a circle.

Diameters can be used to cut a circle into identical slices. Notice that as more diameters are added to the circle, the slices become smaller.



2 Find the length of the curved side of each slice.

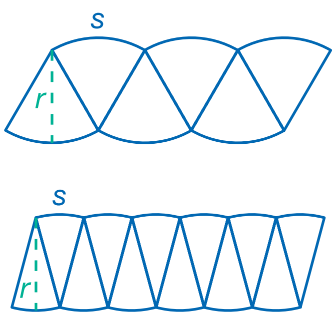
Each curved side is a part of circle. If all of the curved sides were added together, the total would equal the circumference of the circle.

If every slice is identical, each curved side will be the same length. Call this length s. This length can be found by dividing the circle's circumference by the number of slices.

$s = \frac{C}{n} = \frac{2\pi r}{n}$

3 Rearrange the slices into a parallelogram-like shape.

Reassembling the slices into a shape like a parallelogram allows you to use the formula $A = bh$ to approximate their total area.



Assembling smaller sectors makes the shape look more like a parallelogram.

4 Derive the formula for the area of a circle.

The length along the top of the 12-slice figure consists of half of the slices' arcs.

So, if the circle is divided into n slices, $\frac{n}{2}$ curves make up the base of the figure.

base $\approx \frac{n}{2} \cdot s = \frac{n}{2} \cdot \frac{2\pi r}{n} = \pi r$

The height from the top of the figure to the bottom is equal to the radius of the circle.

height $= r$

Multiply these two expressions to derive a formula for the area of the figure.

Area $=$ base $\cdot$ height $\approx \pi r \cdot r = \pi r^2$

This is the formula for the area of a circle.

DISCUSS

If the circle were cut into 100 slices, would the assembled figure look more like a parallelogram or less?

EXAMPLE A Use regular polygons drawn in a circle to derive the formula for the area of a circle in terms of the radius, r.

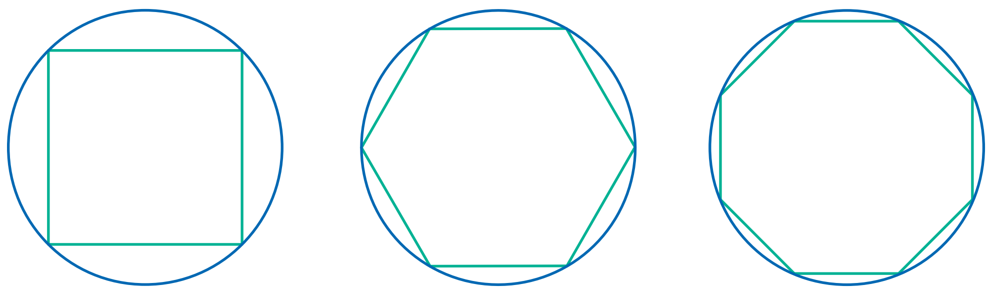
1

Draw polygons into congruent circles.

A polygon with many sides is a better approximation of a circle than a polygon with fewer sides. So, you might want to use a polygon with a large number of sides to find the formula.

But a polygon with 1,000 sides, for example, would be difficult to draw and work with, and it still would not produce the exact formula for a circle.

Instead, draw several figures in congruent circles and see how measurements and calculations change as the number of sides increases.

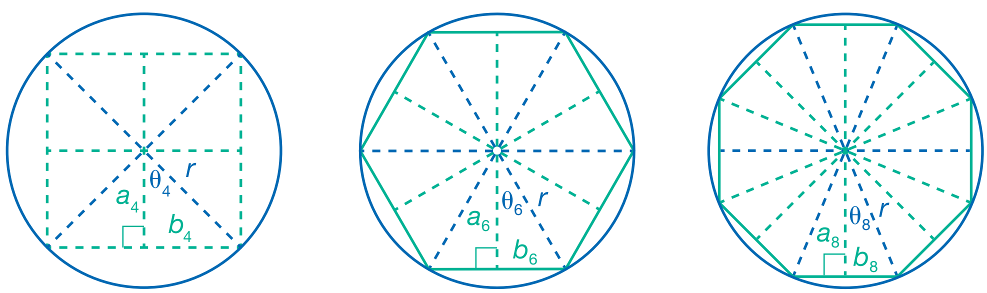


2

Divide the polygons into congruent triangles.

Drawing radii to the vertices of each polygon divides the figure into congruent isosceles triangles. Drawing in the apothems divides them further into congruent right triangles.

The square has 4 sides and is divided into 8 right triangles. The hexagon has 6 sides and is divided into 12 right triangles. The octagon has 8 sides and is divided into 16 right triangles. So, a polygon with n sides would be divided into $2n$ right triangles.



Label the radii, r, and apothems, a, in each figure.

In each figure, the triangles are congruent, so the angles formed by each apothem and radius are congruent. Let θ represent the measure of these angles in each figure.

In each triangle, the leg opposite θ is half of the length of the side of the polygon. Label this length b in each figure.

3 Find an expression for the perimeter of the polygon.

Each side of the polygon has a length of $2b$.

The perimeter of the square is $4 \cdot 2b_4$.

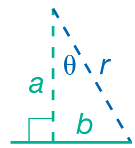
The perimeter of the hexagon is $6 \cdot 2b_6$.

The perimeter of the octagon is $8 \cdot 2b_8$.

In general, the perimeter of a polygon with n sides is $n \cdot 2b$, or $P = 2nb$.

4 Find an expression for a.

Consider one of the right triangles. The hypotenuse is the length of the radius, r.



The side adjacent to θ has a length of a. Relate them with the cosine function.

$$\cos \theta = \frac{a}{r}$$

$$a = r \cos \theta$$

5 Find the polygon's area.

The area of one of the right triangles is

$$A_{\text{tri}} = \frac{1}{2}ab = \frac{1}{2}(r \cos \theta)b$$

A regular polygon with n sides contains $2n$ triangles, so its area is:

$$A_{\text{poly}} = 2n \cdot A_{\text{tri}}$$

$$A_{\text{poly}} = 2n \cdot \frac{1}{2}br \cos \theta$$

$$A_{\text{poly}} = \frac{1}{2}(2nb)r \cos \theta$$

Notice that $2nb$ is the expression for the perimeter of the polygon.

6 Consider what happens as n increases.

As n increases, the polygon looks more and more like a circle. When n is very large, the perimeter of the polygon will be approximately equal to the circumference of the circle. So, replace $2nb$ in the equation with $2\pi r$.

Notice that as the number of sides, n, increases, angle θ becomes smaller. When n is very large, θ will be approximately $0°$. So, replace θ with $0°$.

$$A = \frac{1}{2}(2\pi r)r \cos(0°) = \pi r \cdot r(1)$$

$$A = \pi r^2$$

This formula gives the area of the circle.

CHECK

There are 360° in a circle, so $\theta = \frac{360°}{2n}$.

Use your calculator to check that $\frac{360°}{2n}$ gets closer to $0°$ as n increases.

EXAMPLE B Suppose that a polygon is drawn within a circle having a radius of 1 meter. Find the difference between the circumference of the circle and the perimeter of the polygon if the polygon has 10 sides, 100 sides, or 1,000 sides.

1 Find the circumference of the circle

Remember that the diameter is twice the length of the radius, $d = 2r$.

$C = \pi d$

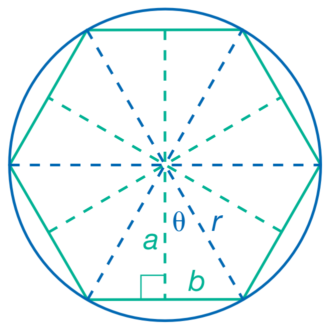
$C = \pi(2r)$

$C \approx 3.14159 \cdot 2 \cdot (1\text{ m})$

$C \approx 6.28318\text{ m}$

2 Find an expression for the perimeter of the polygon.

The same technique of dividing the polygon into right triangles can be used in this problem.



Each side of the polygon is equal to $2b$.

Recall that the length $b = r \sin \theta$. There are a total of 360° in a circle, and the circle is divided into $2n$ right triangles, where n is the number of sides, so $\theta = \frac{360°}{2n}$.

So, the length, l, of a side of the polygon is given by:

$l = 2b = 2r \sin \theta = 2r \sin \frac{360°}{2n}$

The perimeter of the polygon is equal to the number of sides times the length of a side.

$P = nl = 2nr \sin \frac{360°}{2n}$

3 Find the perimeter of each polygon and the difference between its perimeter and the circle's circumference.

Remember that $r = 1$ meter. Use the formula for perimeter and let n equal 10, 100, and 1,000. Then subtract the perimeter values from the approximate circumference of the circle.

n	P	$C - P$
10	6.18034 m	0.10284 m
100	6.28215 m	0.00103 m
1,000	6.28317 m	0.00001 m

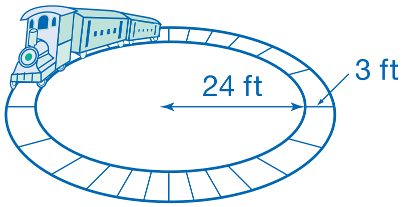
TRY

Repeat these calculations for a circle with a diameter of 1 meter.

Problem Solving

READ

Ariana has a weekend job driving the children's train ride at the amusement park. The two rails of the track form circles with the same center. The radius of the inner circle is 24 feet. The width of the track is 3 feet. In one lap, approximately how much farther does a wheel on the outer track travel than a wheel on the inner track?



PLAN

The distance that each wheel travels is equal to the length around its circular track.

To find how much farther a wheel on the outer track travels than a wheel on the inner track, find the ________________ of each circle and subtract.

SOLVE

Find the exact circumferences in terms of π.

The radius of the inner circle is 24 ft.

C of inner circle $= 2\pi r = 2\pi($______$) =$ ________________

The radius of the outer circle is: ______ $+ 3 =$ ______ ft.

C of outer circle $= 2\pi r = 2\pi($______$) =$ ________________

The difference is: ______ $-$ ______ $=$ ______

Substitute 3.14 for π and approximate the difference: ______ $\cdot\ 3.14 \approx$ ________________

CHECK

All circles are similar, so the ratio of the radius of the inner track to the radius of the outer track should equal the ratio of the ________________ of the inner track to the ________________ of the outer track.

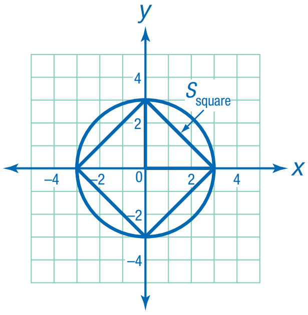
$\frac{24}{27} =$ ______

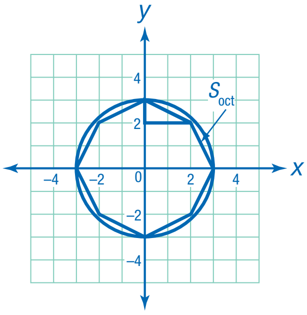
Is this proportion a true number sentence? ______

▶ A wheel on the outer track travels approximately ______ more feet than a wheel on the inner track.

Practice

The circles below have radii of 3 units. A square is drawn within the first circle. An octagon is drawn within the second circle. Use these figures for questions 1–5.





1. Find the exact side length of the square. Estimate its perimeter to the nearest hundredth.

$s_{square} =$ ______________

$P_{square} = 4 \cdot s_{square} \approx$ ______________

HINT Use the Pythagorean Theorem or a trigonometric ratio to find s_{square}.

2. Find the exact side length of the octagon. Estimate its perimeter to the nearest hundredth.

$s_{oct} =$ ______________

$P_{oct} = 8 \cdot s_{oct} \approx$ ______________

HINT The hypotenuse of the small right triangle is equal to s_{oct}.

3. The circle's circumference is greater than the square's perimeter. Complete the inequality.

$2\pi r > P_{square}$

2π (______________) > ______________

$\pi >$ ______________

4. The circle's circumference is greater than the octagon's perimeter. Complete the inequality.

$2\pi r > P_{oct}$

2π(___) > ______________

$\pi >$ ______________

5. Suppose you keep increasing the number of sides of the inscribed polygon. What will happen to your approximation of π?

____

Find the circumference and area of circles with the given dimensions.

6. a radius of 4 inches

Circumference: ______________

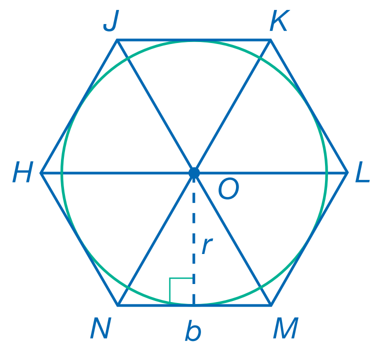
Area: ______________

7. a diameter of 10 centimeters

Circumference: ______________

Area: ______________

Polygon *HJKLMN* is drawn around circle *O*. Use this figure for questions 8–11.



8. What is the perimeter P of a polygon with n sides if each side has a length of b units?

P = ____________

9. Write an expression for the area of $\triangle MNO$ in terms of b and r.

____

10. A polygon with n sides could be broken into n such triangles. What would be the total area of that polygon, in terms of n, b, and r?

____

What is the polygon's area in terms of its perimeter P and r?

____

11. For very large values of n, the perimeter of the polygon would be about equal to the circumference of the circle. Substitute $2\pi r$ for the P in your equation for the polygon's area and simplify.

____

____

Solve.

12. ILLUSTRATE Julissa's bedroom floor is 12 feet by 12 feet. She places a circular rug on her floor so that it touches each side of the room. Approximately how many square feet of her bedroom floor will **not** be covered by the rug? Give an exact answer and an approximate answer to the nearest square foot.

____

____

13. ANALYZE A circle has a circumference of 5π meters. What is its area? ____________